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1987 J. Phys. A: Math. Gen. 20 L595

(http://iopscience.iop.org/0305-4470/20/9/008)

J. Phys. A: Math. Gen. **20** (1987) L595–L600. Printed in the UK

LETTER TO THE EDITOR

Finite-size scaling study of two-dimensional dilute Potts models

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Received 9 April 1987

Abstract. We use transfer matrices to calculate the free energy and various thermodynamic quantities of the two- and three-state Potts models with random bond interactions at the critical temperature. We verify the Harris criterion for the three-state Potts model and estimate the new random critical exponents. For the Ising case, our results are consistent with randomness being irrelevant for the critical behaviour.

Magnetic systems with quenched random impurities have become the subject of much study recently. One generally distinguishes between randomness in the interaction strength and in the magnetic field. The latter has usually a drastic effect on the critical behaviour of a system. In this letter we report the results of a calculation done on the random bond two- and three-state Potts model in two dimensions. These may be relevant for describing order–disorder transitions in adsorbed monolayers in the presence of substrate imperfections and frozen impurities. Harris (1974) first predicted that bond randomness causes crossover from the pure to a new (random) critical behaviour provided that the pure specific heat exponent $\alpha_P > 0$. Since $\alpha_P = \frac{1}{3}$ for the three-state Potts model (den Nijs 1979, Alexander 1975, Baxter 1980), it is interesting to investigate the effect of bond randomness in this case. Some preliminary results have been obtained by Kinzel and Domany (1981) and Yeomans and Stinchcombe (1980) by means of a real space renormalisation group calculation.

For the Ising (two-state Potts) model, $\alpha_P = 0$ (Onsager 1944) and the Harris criterion cannot be used, but one would expect any pure system with a second-order phase transition characterised by a divergent specific heat to be strongly sensitive to bond impurities in the critical region.

Early Monte Carlo simulations on the two-dimensional random bond Ising model have shown no changes from the pure critical behaviour (Ching and Huber 1976, Stoll and Schneider 1976, Fisch and Harris 1976). In addition, in a neutron scattering experiment on the site diluted two-dimensional antiferromagnet $Rb_2Co_{0.7}Mg_{0.3}F_4$, the correlation length and staggered susceptibility exhibited pure Ising critical behaviour above the Néel temperature (Birgeneau *et al* 1983). It seems then that bond randomness is irrelevant in the Ising model.

However, Dotsenko and Dotsenko (1983) arrive at a different conclusion. They start with the well known formulation of the 2D Ising model in the critical region as a continuum free fermion field theory. In analogy to the replica formulation the impurities are then shown to give rise to a four-fermion interaction in an n-component field, whose strength is proportional to the impurity concentration. These models are known to be renormalisable in two dimensions. For the relevant $n \to 0$ limit, the critical behaviour could be obtained exactly by the renormalisation group method. They found

0305-4470/87/090595+06$02.50

that for any small concentration p of impurities the specific heat C changes from its pure $C \sim -\ln|t|$ behaviour $(t=(T-T_c)/T_c)$ to $C \sim -\ln|\ln|t||$ provided $t \ll \exp(-\text{constant}/p)$; more drastically for the two-point function $\langle S_0; S_R \rangle$, which decays as $R^{-\eta}$ with $\eta = \frac{1}{4}$ for the pure Ising model at criticality, η changes to zero provided $R \gg \exp(\text{constant}/p)$. It can be expected that these properties hold for any $p \neq 0$.

Very recently, Ludwig (1986a) performed a renormalisation group calculation on the two-dimensional random q-state Potts model, where, in analogy to the $(4-d)$-expansion for ϕ^4-field theory around the Gaussian model, a $(q-2)$-expansion around the two-dimensional Ising model is carried out. He confirms the ln-ln form of the specific heat for the random Ising model and obtains the leading corrections to it. His calculations, among other things, also yield the value

$$y_T = 0.98 \tag{1}$$

for the random thermal exponent of the three-state Potts model.

The system studied in the following consists of Potts spins $\sigma_i \in \{1, \ldots, q\}$ placed on the sites i of a square lattice. To each spin configuration is attributed an energy

$$H = \sum_{\langle ij \rangle} J_{\langle ij \rangle} \delta_{\sigma_i, \sigma_j} \tag{2}$$

where $\langle ij \rangle$ denotes nearest neighbours, $\delta_{\sigma_i,\sigma_j}$ is the Kronecker symbol and the $J_{\langle ij \rangle}$ are independent random variables, each one having the probability distribution

$$p(J) = \tfrac{1}{2}\{\delta(J+1) + \delta(J+\varepsilon)\} \tag{3}$$

with $0 \leqslant \varepsilon \leqslant 1$. For this distribution, the critical temperature $T_c = \beta_c^{-1}$ is given exactly (Fisch 1978, Kinzel and Domany 1981) by the equation

$$(e^{\beta_c} - 1)(e^{\varepsilon\beta_c} - 1) = q. \tag{4}$$

From now on we fix $\varepsilon = \frac{1}{2}$, corresponding to $T_c \approx 0.82$ for $q=2$ and $T_c \approx 0.72$ for $q=3$.

We have evaluated the free energy and its temperature (T) and magnetic field (h) derivatives on long strips of width N with $2 \leqslant N \leqslant 8$ for $q=2$ and $2 \leqslant N \leqslant 7$ for $q=3$.

Let

$$H_l = \sum_{n=1}^{N} \{J_{l,n;l,n+1} \delta_{\sigma_{l,n},\sigma_{l,n+1}} + J_{l,n;l+1,n} \delta_{\sigma_{l,n},\sigma_{l+1,n}}\} \tag{5}$$

denote the Hamiltonian in row l along the strip with periodic boundary conditions across the strip. The transfer matrix

$$T_l = \exp(-\beta_c H_l) \tag{6}$$

acts on the q^N-dimensional space V of spin configurations. Starting with any vector v_0 (with length $\|v_0\|$) in V and any bond realisation, the random bond configurational averaged free energy per site F_{av}^N on an infinitely long strip is given by (Furstenberg 1963)

$$F_{av}^N = -(\beta_c N)^{-1} \Lambda_{0,N} \tag{7a}$$

$$\Lambda_{0,N} = \lim_{L\to\infty} \frac{1}{L} \ln\left\{\left(\left\|\prod_{l=1}^{L} T_l v_0\right\|\right)(\|v_0\|)^{-1}\right\}. \tag{7b}$$

In practice, L has to be chosen large enough to ensure a small statistical error. If the T and h derivatives of F_{av}^N were taken numerically, these statistical uncertainties would be enhanced by a few orders of magnitude for each derivative resulting in useless data. Fortunately, F_{av}^N is an analytic function of T and h. It is therefore possible to simultaneously iterate several vectors corresponding to the coefficients of the Taylor expansion in powers of T and h. To obtain the correlation length along the strips, a vector v_1 in the subspace corresponding to the appropriate irreducible representation of the symmetric group s_q has to be iterated yielding $\Lambda_{1,N}$ as in (7*b*). The correlation length is then given by

$$\xi_N^{-1} = \ln(\Lambda_{0,N}/\Lambda_{1,N}). \qquad (8)$$

We have evaluated ξ_N at T_c for $q = 2, 3$. As expected from finite-size scaling (Fisher 1971), ξ_N is found to be approximately proportional to N. For $q = 3$, we estimate the exponent ν defined by

$$\xi \sim |T - T_c|^{-\nu} \qquad (9)$$

where ξ is the correlation length in the thermodynamic limit. For two successive strip widths, $\nu_{N,N+1}$ follows from the finite-size scaling relation

$$1 + \nu_{N,N+1}^{-1} = \ln(\xi_{N+1}^1/\xi_N^1)/\ln((N+1)/N) \qquad (10)$$

where ξ_N^1 denotes the T derivative of ξ_N at T_c. Figure 1 shows a plot of $\nu_{N,N+1}$ against N^{-1} for the random three-state Potts model together with the exact values for the pure (non-random) case for comparison. The strip lengths L were $L = 5 \times 10^5$ for $2 \leqslant N \leqslant 5$, $L = 1.5 \times 10^5$ for $N = 6$ and $L = 5 \times 10^4$ for $N = 7$. The error bars in all figures were estimated from observing the fluctuations in $\Lambda_{0,N}$ and $\Lambda_{1,N}$ and their temperature derivatives by recording their values every 200 transfer matrix iterations for three independent runs for each value of N. It is obvious from this plot that already for strips as small as we have considered, the values for $\nu_{N,N+1}$ are distinctly larger for the random system than in the pure case. We estimate

$$\nu = 1.05 \pm 0.1. \qquad (11)$$

This should be compared with equation (1) where $\nu = 1/y_T$.

At critical points in 2D models, the correlation functions are invariant under conformal transformations. It was shown (Cardy 1984) that this implies for the correlation length on the strip

$$\xi_N = N/(\pi\eta) \qquad (12)$$

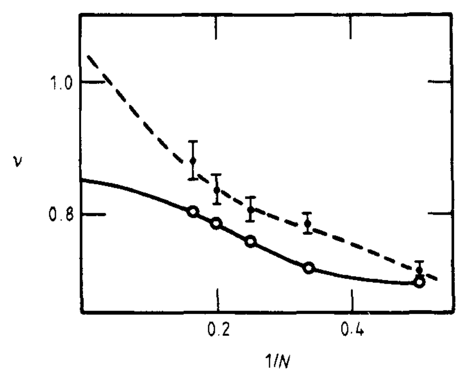

Figure 1. Estimates of $\nu_{N,N+1}$ (equation (10)) against $1/N$ for $q = 3$. Open circles denote the pure and full circles the random model values. Error bars in the random case were obtained as explained in the text. Broken and full curves are guides to the eye used for the extrapolation $N \to \infty$.

where η describes the power law decay of the two-point correlation function. In figure 2 we plot η_N from (12) for (*a*) $q=2$ and (*b*) $q=3$ for the random and pure systems against N^{-1}. For $q=2$, $L=2\times10^5$ for all N. Figure 2(*a*) shows that for $q=2$, the pure and random values for η_N behave very similarly for the strip widths considered. In fact, a naive extrapolation of the random data suggests

$$\eta = 0.25 \pm 0.01 \tag{13}$$

in contradiction to the exact $\eta=0$ (Dotsenko and Dotsenko 1983). It seems that crossover to the predicted random behaviour occurs on a much larger length scale and could therefore only be observed if N was increased by an order of magnitude. For $q=3$ (figure 2(*b*)) the random values deviate rather strongly from the pure ones and we estimate

$$\eta = 0.285 \pm 0.01 \tag{14}$$

for the random three-state Potts model, distinctly larger than the exact conjectured pure value $\eta=0.267$ (Nienhuis *et al* 1980, Pearson 1980, Baxter 1980).

For $q=2$ we have also calculated the magnetic susceptibility χ_N by applying a uniform field h giving rise to a term

$$h\sum_i \delta_{\sigma_i,1} \tag{15}$$

to be added to the Hamiltonian defined in (2). χ_N is then given by

$$\chi_N = \left.\frac{\partial^2 F^N_{\mathrm{av}}}{\partial h^2}\right|_{h=0}. \tag{16}$$

From finite-size scaling it is expected that

$$\chi_N \sim N^{\gamma/\nu} \tag{17}$$

at T_c, where γ is defined by $\chi_{N=\infty}(T)\sim|T-T_c|^{-\gamma}$. Assuming $\nu=1$, figure 3 shows a plot of our estimates

$$\gamma_{N,N+1} = \ln(\chi_{N+1}/\chi_N)/\ln((N+1)/N) \tag{18}$$

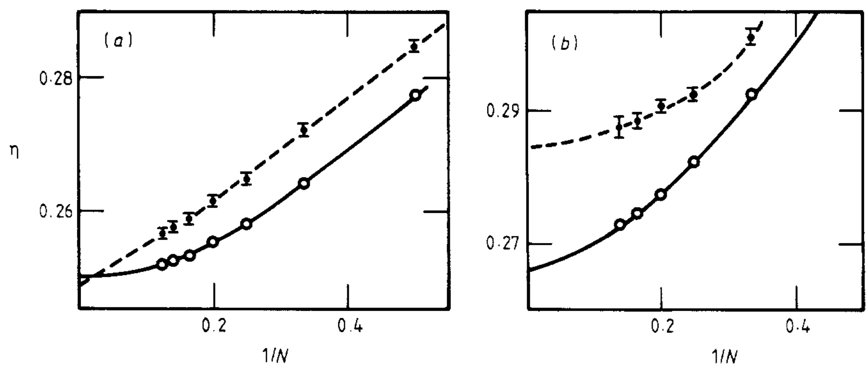

Figure 2. η_N obtained from (12) against $1/N$ for (*a*) $q=2$ and (*b*) $q=3$. The symbols are the same as in figure 1. The continuations of the curves to the upper right-hand corner of figure 2(*b*) point in the direction of the estimated values for $N=2$, not shown in the figure.

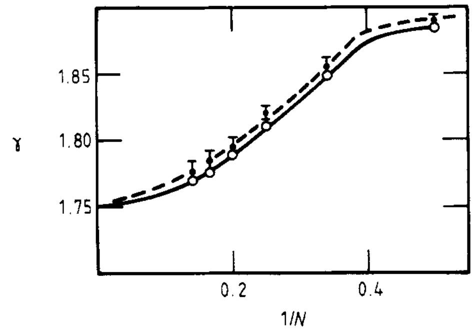

Figure 3. $\gamma_{N,N+1}$ obtained from (18) against $1/N$ for $q=2$. The symbols are the same as in figure 1.

against N^{-1} for $2 \leqslant N \leqslant 7$. The data for the random and pure system are very close and straight extrapolation of the random values yields

$$\gamma = 1.75 \pm 0.03. \tag{19}$$

There is no agreement with the predicted $\gamma = 2$ (Dotsenko and Dotsenko 1983).

Blöte *et al* (1986) and Affleck (1986) have shown that from the scaled strip free energy $\beta_c F_{av}^N$ at T_c one can estimate the conformal anomaly number c by using

$$\beta_c F_{av}^N = \beta_c F_{av}^\infty - \frac{\pi}{6N^2} c + O(1/N^2). \tag{20}$$

Recently, Ludwig (1986b) has calculated the leading $O(1/N^2)$ corrections for $q=2$ in (20) and shown that, contrary to the pure case, the effective $c(N, N+1)$ as obtained from (20) by comparing two successive strip widths neglecting corrections should converge to its asymptotic value $c = \frac{1}{2}$ from below. We have tried to verify this behaviour by producing high-quality data for $\Lambda_{0,N}$, which are given in table 1 together with the resulting estimates for $c(N, N+1)$. As in the pure case, the values seem to converge to $\frac{1}{2}$ from above in disagreement with the renormalisation group calculation.

Our results for the critical exponents and the corrections to scaling in the conformal anomaly for the $q=2$ state Potts model (or the Ising model) with random interactions are in disagreement with analytical calculations by Dotsenko and Dotsenko (1983) and Ludwig (1986b). A natural explanation for this, already provided by Dotsenko

Table 1. Scaled free energy $-\beta F$ and resulting estimate for the conformal anomaly $c(N, N+1)$ obtained from (20) by neglecting $O(N^{-2})$ corrections. N denotes the strip width and L the strip length. The number in brackets for $-\beta F$ denotes uncertainty in the last quoted digits. Error bars for c were obtained directly from those of $-\beta F$.

N	L	$-\beta F$	c
2	10^5	1.039 43 (20)	
			0.605 + 0.005
3	2×10^5	0.995 55 (13)	
			0.55 + 0.01
4	10^6	0.981 63 (10)	
			0.56 + 0.02
5	10^6	0.975 04 (8)	
			0.51 + 0.03
6	10^6	0.971 75 (7)	
			0.54 + 0.04
7	2×10^6	0.969 66 (6)	
			0.50 + 0.05
8	2×10^6	0.968 40 (5)	

and Dotsenko, is that crossover to random critical behaviour only sets in on very large length scales or for very large strip width N. The probability distribution (3) for the random bonds chosen for our calculation implies that the concentration p of impurities is very large. If the estimate $R_c \sim \exp(\text{constant}/p)$ for the crossover length scale applies also for our distribution, R_c would be of the order of a few lattice spacings and crossover effects should be seen with the strip widths we considered. If, as seems to be the case, R_c is much larger, it is hard to find a simple physical reason for the existence for it. A theory for the crossover behaviour in the random bond Ising model would therefore be very useful.

For $q=3$, we have found strong evidence for the validity of the Harris criterion and provided estimates for the random exponents ν and η. While we do not know of any other calculation of η in the literature, the estimate for ν agrees well with the renormalisation group calculation of Ludwig (1986a).

I would like to thank N Bartelt for very helpful discussions. This work was partially supported by a grant from the US Department of Energy under Grant no DE-FG 05-84ER45071.

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